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THE IMPLICIT DIFFERENCE SCHEME FOR NUMERICAL SOLVING THE EQUATIONS OF GAS DYNAMICS

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Abstract—Economical difference schemes based on splitting in terms of the physical processes and the space variables involved are proposed for numerical solving the equations of gas dynamics and Navier–Stokes equations. The schemes are operated by using the moving grids which adapt themselves automatically to the solution that allow to increase essentially the computation accuracy. An example of computation is presented.

Main requirements to the finite-difference schemes for solving the equations of gas dynamics and Navier–Stokes' equations may be formulated as follows: the difference scheme should possess a sufficient accuracy and be economical and simple in operating.

Explicit difference schemes[1–3] though simple in operating are uneconomical in view of the rigid restrictions imposed on the stability, for example, both when solving Navier–Stokes' equations at moderate and low Reynolds numbers and obtaining the stationary solution by the relaxation method. Implicit difference schemes[4–6] permit to relax the requirements imposed on the stability reducing the solution to the condition of CFL type. The schemes[7–9] are absolutely stable. Implicit schemes are more complex in operating, however the application of the splitting-up method[10] enables one to simplify the algorithm of solution.

The increase in the calculation accuracy may be achieved either when employing the higher order schemes[11, 12] or utilizing the non-uniform grids. The first approach leads, as a rule, to the algorithm complication. It is necessary to have *a priori* knowledge on the solution to employ effectively the non-uniform grids. Often the information of such a kind is not available especially for complex flows and computed regions. Therefore it is expedient to employ the non-uniform grids adapting themselves automatically to the solution. The present approach is developed, for instance, in papers[11, 13].

In the present paper, the implicit difference scheme of splitting in terms of the physical processes and the space variables involved which possess the properties of full approximation and absolute stability is proposed for numerical solving the equations of gas dynamics and Navier–Stokes' equations. The scheme is operated by scalar sweeps which render it economical. To increase the accuracy of calculations, the moving difference grid depending on the gradients of solution[13] is employed.

GOVERNING EQUATIONS

For simplicity of the statement we shall consider the system of two-dimensional equations in cartesian coordinates. The Navier–Stokes equations (at $\mu \equiv 0$—the equations of gas dynamics) can be presented in the vector form

$$\frac{\partial \mathbf{F}}{\partial t} = -\mathbf{W} = -\left(\frac{\partial}{\partial x}\mathbf{W}_1 + \frac{\partial}{\partial y}\mathbf{W}_2\right), \tag{1}$$

where

$$\mathbf{F} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \quad \mathbf{W}_1 = \begin{pmatrix} \rho u \\ \rho u^2 + p - G_{xx} \\ \rho uv - G_{xy} \\ u(E+p) - \lambda(\partial T/\partial x) - uG_{xx} - vG_{xy} \end{pmatrix},$$

$$\mathbf{W}_2 = \begin{pmatrix} \rho v \\ \rho uv - G_{xy} \\ \rho v^2 + p - G_{yy} \\ v(E+p) - \lambda(\partial T/\partial y) - uG_{xy} - vG_{yy} \end{pmatrix}$$

$$G_{xx} = 2\mu \frac{\partial u}{\partial x} + \left(\zeta' - \frac{2}{3}\mu\right) \operatorname{div} \mathbf{v}, \quad G_{yy} = 2\mu \frac{\partial v}{\partial y} + \left(\zeta' - \frac{2}{3}\mu\right) \operatorname{div} \mathbf{v},$$

$$G_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \operatorname{div} \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}.$$

The following nomenclature is accepted in equations: x, y are the Cartesian coordinates, respectively, u, v is the projection of the velocity vector $\mathbf{v}$ on the coordinate axis, p is the pressure, T is the temperature, $E = \rho(\epsilon + (u^2 + v^2/2))$ is the total energy, ϵ is the internal energy, μ and ζ' are the first and second coefficients of viscosity, λ is the coefficient of thermal conductivity.

To close the system of equations (1), we set the equation of state

$$p = p(\rho, \epsilon), \tag{2}$$

the law of dependence of viscosity coefficients and thermal conductivity on temperature

$$\mu = \mu(T), \quad \zeta' = \zeta'(T), \quad \lambda = \lambda(T) \tag{3}$$

and the connection between internal energy and temperature. The solution of the system of equations (1)–(3) is sought in the region D with the appropriate boundary conditions.

Assume that the sought functions change sharply along one of the coordinates. For example, if the axis x is directed along the body surface, then the main change of functions will be in the transversal direction y. Let us introduce the nondegenerate transformation of coordinates

$$\xi = \xi(x), \quad \eta = \eta(x, y) \tag{4}$$

transforming the computed region D into a single square $R\{0 \le \xi \le 1, 0 \le \eta \le 1\}$. The reverse transformation is set by relations

$$x = x(\xi), \quad y = y(\xi, \eta). \tag{5}$$

It is well-known[14] that the system of equations in ξ and η variables can be presented in the divergent form:

$$\frac{\partial}{\partial t} \Delta \mathbf{F} = -\mathbf{W} = -\left[\frac{\partial}{\partial \xi} \Delta z_0 \mathbf{W}_1 + \frac{\partial}{\partial \eta} \Delta(z\mathbf{W}_1 + z_1 \mathbf{W}_2)\right], \tag{6}$$

where

$$\Delta = \frac{\partial(x, y)}{\partial(\xi, \eta)}, \quad z_0 = \frac{\partial \xi}{\partial x} = 1 \Big/ \frac{\partial x}{\partial \xi}, \quad z_1 = \frac{\partial \eta}{\partial y} = 1 \Big/ \frac{\partial y}{\partial \eta}, \quad z = \frac{\partial \eta}{\partial x} = -z_0 z_1 \frac{\partial y}{\partial \xi}.$$

Choosing the vector $\mathbf{f}$ with components ρ, u, v, ϵ as the unknown functions, we present the system of equations (6) in the non-divergent form in the form of splitting in terms of space directions and physical processes[8]:

$$\frac{\partial \mathbf{f}}{\partial t} = -\sum_{j=1}^{4} \Omega_j \mathbf{f} + \mathbf{H}. \tag{7}$$

Differential matrix operators Ω_1 and Ω_2 take into account the convective and viscous terms (the mixed derivatives are ignored) in the direction of ξ and η, respectively, Ω_3 and Ω_4 are the terms with pressure and the terms of the form div v in equations of continuity and energy in each direction can be presented in the form:

$$\mathbf{f} = \begin{pmatrix} \rho \\ u \\ v \\ \epsilon \end{pmatrix}, \quad \Omega_1 \mathbf{f} = z_0 u \frac{\partial}{\partial \xi} \mathbf{f} - \frac{z_0}{\rho} \frac{\partial}{\partial \xi} z_0 \begin{pmatrix} 0 \\ \left(\frac{4}{3}\mu + \zeta'\right) \frac{\partial u}{\partial \xi} \\ \mu \frac{\partial v}{\partial \xi} \\ \frac{\gamma \mu}{\mathrm{Pr}} \cdot \frac{\partial \epsilon}{\partial \xi} \end{pmatrix}$$

$$\Omega_2 \mathbf{f} = (zu + z_1 v) \frac{\partial}{\partial \eta} \mathbf{f} - \frac{z}{\rho} \cdot \frac{\partial}{\partial \eta} z \begin{pmatrix} 0 \\ \left(\frac{4}{3}\mu + \zeta'\right) \frac{\partial u}{\partial \eta} \\ \mu \frac{\partial v}{\partial \eta} \\ \frac{\gamma \mu}{\mathrm{Pr}} \frac{\partial \epsilon}{\partial \eta} \end{pmatrix} - \frac{z_1}{\rho} \frac{\partial}{\partial \eta} z_1 \begin{pmatrix} 0 \\ \mu \frac{\partial u}{\partial \eta} \\ \left(\frac{4}{3}\mu + \zeta'\right) \frac{\partial v}{\partial \eta} \\ \frac{\gamma \mu}{\mathrm{Pr}} \frac{\partial \epsilon}{\partial \eta} \end{pmatrix}$$

$$\Omega_3 \mathbf{f} = z_0 \begin{pmatrix} \rho \frac{\partial u}{\partial \xi} \\ a^2 \frac{\partial \rho}{\partial \xi} + b^2 \frac{\partial \epsilon}{\partial \xi} \\ 0 \\ c^2 \frac{\partial u}{\partial \xi} \end{pmatrix}, \quad \Omega_4 \mathbf{f} = \begin{pmatrix} \rho \left[z \frac{\partial u}{\partial \eta} + z_1 \frac{\partial v}{\partial \eta} \right] \\ z \left[a^2 \frac{\partial \rho}{\partial \eta} + b^2 \frac{\partial \epsilon}{\partial \eta} \right] \\ z_1 \left[a^2 \frac{\partial \rho}{\partial \eta} + b^2 \frac{\partial \epsilon}{\partial \eta} \right] \\ c^2 \left[z \frac{\partial u}{\partial \eta} + z_1 \frac{\partial v}{\partial \eta} \right] \end{pmatrix},$$

$$a^2 = \frac{1}{\rho} \frac{\partial p}{\partial \rho}, \quad b^2 = \frac{1}{\rho} \frac{\partial p}{\partial \epsilon}, \quad c^2 = p/\rho.$$

The vector **H** contains the mixed derivatives $\gamma = c_P/c_V$, $\lambda = c_P \mu/\mathrm{Pr}$ (γ and Pr are supposed to be constant). The temperature is eliminated from the equations with the help of formula $\epsilon = c_V T$. The system of equations (4) can be presented in the form:

$$\frac{\partial \mathbf{f}}{\partial t} = -\frac{1}{\Delta} B \mathbf{W}, \tag{8}$$

where

$$\Delta \neq 0, \quad B = \left(\frac{\partial \mathbf{F}}{\partial \mathbf{f}}\right)^{-1} = \frac{1}{\rho} \begin{pmatrix} \rho & 0 & 0 & 0 \\ -u & 1 & 0 & 0 \\ -v & 0 & 1 & 0 \\ -\left(\epsilon - \frac{u^2+v^2}{2}\right) & -u & -v & 1 \end{pmatrix}$$

The equations (6)–(8) are equivalent in the differential form.

We shall set the transformation of coordinates (4) so that the mesh points of the difference grid should condense in a region of large gradients, i.e. we require the condition

$$\left(\left| \frac{\partial \mathbf{f}}{\partial y} \right|^{\alpha} + \delta \right) \frac{\partial y}{\partial \eta} = C = \mathrm{Const} \tag{9}$$

to be implemented. The condition of the type (9) is obtained in the paper[13] when constructing the moving grids on the basis of variational principle. A necessary condensation of mesh points of the difference grid is set by choice of the coefficient α. The value of y-coordinate may be found from (8) or as in the paper[13] from the solution of equation

$$\frac{\partial y}{\partial t_1} = \frac{\partial}{\partial \eta}\left(\left|\frac{\partial \mathbf{f}}{\partial y}\right|^{\alpha} + \delta\right)\frac{\partial y}{\partial \eta} \tag{10}$$

with the stationary boundary-value conditions. The x-coordinate may be found from (5) or from the equation of the type (9); (10). The coefficients of transformation z_0, z_1, z and Δ are found after the $x = x(\xi)$ and $y = y(\xi, \eta)$-coordinates are obtained.

THE DIFFERENCE SCHEME

We introduce the difference grid with the space steps h_p ($h_p = (1/T_p)$, $p = 1, 2$, T_p is the number of mesh points in the direction p) and with the time step τ in $R_H = R \times H$. We determine the grid functions $\mathbf{f}_h{}^n$ approximating the functions $\mathbf{f}$ in the mesh points of the grid. To approximate the both first derivatives $(\partial/\partial\xi)$ and $(\partial/\partial\eta)$, we employ the non-symmetric approximation $\Lambda^k_{\mp p}$ with the order (the sign + or − is chosen depending on the velocity sign):

$$\Lambda^k_{\mp p} g_{ip} = \begin{cases} \Lambda^1_{\mp p} g_{ip} = \pm \dfrac{1}{h_p}(g_{ip} - g_{ip\mp 1}) \\[2ex] \Lambda^2_{\mp p} g_{ip} = \pm \dfrac{1}{2h_p}(3g_{ip} - 4g_{ip\mp 1} + g_{ip\mp 2}) \end{cases}$$

and we approximate the second derivatives in the similar way by symmetric operators on the three-point or nine-point mesh element. Then we approximate the differential matrix operators Ω by difference operators $\Omega_j{}^k$ with the order k ($k = 1, 2$). The form of both difference operators $\Omega_1{}^k$ and $\Omega_3{}^k$ is presented as an example:

$$\Omega_1{}^k = z_{0h}\left(u_h{}^n\Lambda^k_{\mp 1} - \frac{1}{\rho_h{}^n}\Lambda_1 z_{0h} q_1{}^n \Lambda_1\right)\cdot I, \quad \Lambda_1 d\Lambda_1 \approx \frac{\partial}{\partial\xi} d \frac{\partial}{\partial\xi}$$

$$\Omega_3{}^k = z_{0h}\begin{pmatrix} 0 & \rho_h{}^n\Lambda^k_{\mp 1} & 0 & 0 \\ (a^2)_h{}^n\Lambda^k_{\pm 1} & 0 & 0 & (b^2)_h{}^n\Lambda^k_{\pm 1} \\ 0 & 0 & 0 & 0 \\ 0 & (c^2)_h{}^n\Lambda^k_{\mp 1} & 0 & 0 \end{pmatrix}, \quad q_1 = \begin{pmatrix} 0 & & \\ & \frac{4}{3}\mu + \zeta' & 0 \\ & 0 & \mu \\ & & & \frac{\gamma}{\mathrm{Pr}}\mu \end{pmatrix}.$$

Note that the terms with the pressure $a^2(\partial/\partial\xi)$ and $b^2(\partial/\partial\xi)$ are approximated by the non-symmetric operators $\Lambda^k_{\pm 1}$ using the formulae that are conjugate with respect to convective terms. Both operators $\Omega_2{}^k$ and $\Omega_4{}^k$ are written in the similar way. This approximation enables one to construct a stable difference scheme. The vector $\mathbf{W}$ of the system of eqns (8) is approximated consistently with the operators $\Omega_j{}^k$: the first derivatives—by the non-symmetric differences, and the second derivatives—by the symmetric ones (in the case of one-dimensional equations the difference approximation of eqn (7) is considered in details in [8]).

We consider the difference scheme of splitting in terms of the physical processes and the space variables for numerical solution of the system of eqns (8):

$$
\begin{aligned}
(I + \tau\alpha_1\Omega_1{}^l)\xi^{n+(1/4)} &= -\tau\left(\frac{B}{\Delta}\right)_h^n \mathbf{W}_h{}^k \\
(I + \tau\alpha_2\Omega_2{}^l)\xi^{n+(2/4)} &= \xi^{n+(1/4)} \\
(I + \tau\alpha_3\Omega_3{}^l)\xi^{n+(3/4)} &= \xi^{n+(2/4)} \\
(I + \tau\alpha_4\Omega_4{}^l)\xi^{n+1} &= \xi^{n+(3/4)} \\
\mathbf{f}_h{}^{n+1} &= \mathbf{f}_h{}^n + \xi^{n+1}.
\end{aligned}
\tag{11}
$$

Eliminating the fractional steps in the scheme (11), we obtain the difference scheme of the type of universal algorithm:

$$
C^l \frac{\mathbf{f}^{n+1} - \mathbf{f}^n}{\tau} = -\left(\frac{B}{\Delta}\right)_h^n \mathbf{W}_h{}^k, \quad C^l = \prod_{j=1}^{4} (I + \tau\alpha_j\Omega_j^l) \tag{12}
$$

the scheme approximates the system of equations (8) with the order $0(\tau + h^l + h^k)$ and in the case of relaxation process with the order $0(h^k)$ where $h = \max(h_p)$.

In the case of consistent operator $C^l(l = k)$ the difference scheme is operated by inverting $2l+1$-diagonal matrices. In the process of obtaining the stationary solution by the relaxation method we choose the operator C^1. The difference scheme (12) is operated by inverting triadiagonal matrices and in the case of obtaining the stationary solution it has an error $0(h^k)$, $k = 1, 2$ and is conservative (the matrix B is nonsingular and the stationary equations are written in the form of implementation of the difference conservation laws).

The scheme operating at each fractional step is performed similarly to [7] where the nondivergent analogue of the scheme (12) is considered.

The proof of the difference scheme (11) stability is carried out by Fourier method for the system of linear equations obtained by the linearization of eqns (8). The analysis of stability has shown that the difference scheme (11) is absolutely stable at $\alpha_j \geqslant 0.5$ for the consistent operator C_l and at $\alpha \geqslant 1$ for the nonconsistent operator.

The equation for the grid (9) is solved numerically at each time step until obtaining the stationary solution with the help of the implicit difference scheme

$$
\frac{y_i^{n+1} - y_i^n}{\tau} = \Lambda_2(|\Lambda_i f|^\alpha + \delta)\Lambda_2 y_i^{n+1}. \tag{12}
$$

After obtaining the stationary solution of the equation, the values of Jacobian transformation Δ_h and the coefficients z_{0h}, z_h and z_{1h} with the second order of accuracy are determined numerically.

NUMERICAL CALCULATIONS

Calculations of the flow past the axisymmetric composite bodies by supersonic stream of a viscous compressible gas were carried out with the help of the algorithm suggested. The body flowed around consisted of the cylinder blunted in sphere that transforms in the cone with angle of taper $\theta = 41°46'$ and the cylinder with different radius. The computed region was divided into two subregions Q_1 and Q_2. In the region Q_1 (Fig. 1) the spheric coordinate system was chosen, and in Q_2 (Fig. 3)—the cylindrical one. Calculations were carried out in Q_1 and Q_2. The solution obtained in Q_1 (subregions Q_1 and Q_2 has a common intersection) was taken as boundary-value condition at $x = 0$ in Q_2. Both the no-slip conditions and the cooling conditions were set on the body surface, where $T_w = aT_T$ is the stagnation temperature. The boundaries of the computed region were set so that the perturbations induced by the body and the shock wave the position of which is determined in the process of solution could not reach the external boundary $R_1(x)$, and the undisturbed flow was set on this boundary

$$
\rho_\infty = 1, \quad u_\infty = u_0, \quad v_\infty = v_0, \quad \epsilon_\infty = \frac{1}{\gamma(\gamma - 1)M_\infty^2}.
$$

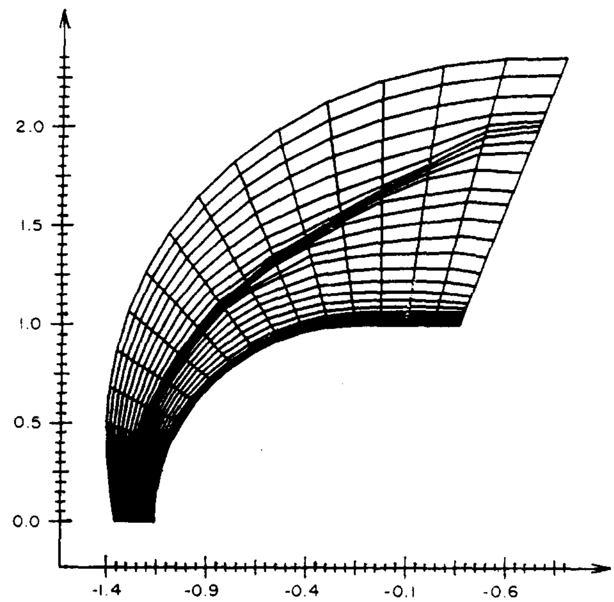

Fig. 1.

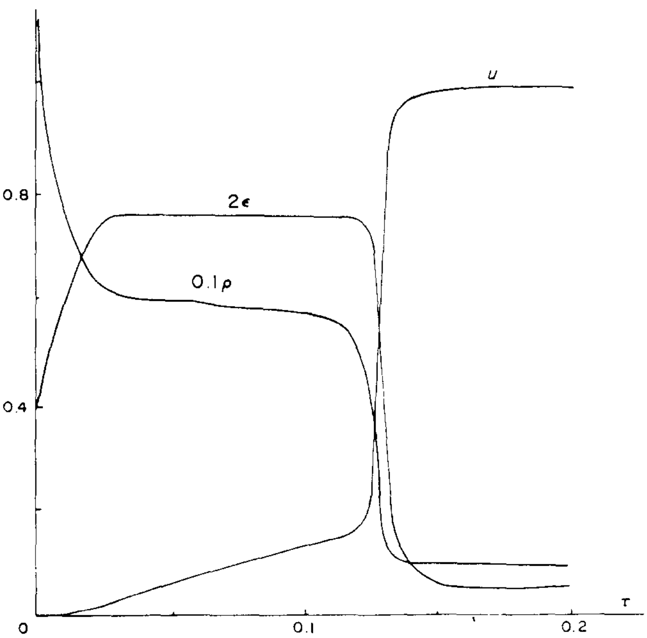

Fig. 2.

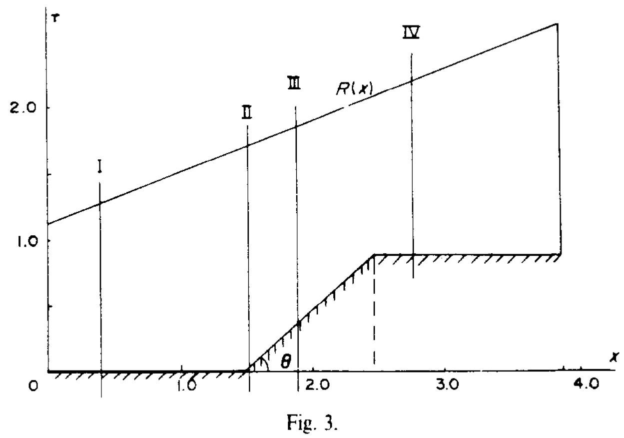

Fig. 3.

"Soft" conditions were set on the trail boundary, i.e. it was assumed

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 \epsilon}{\partial x^2} = 0.$$

It is well-known that the conditions of the type (13) influence weakly the flow upstream. The mesh step in ξ direction was chosen uniform, and in η direction it was determined from the stationary solution of the equation for the grid (12). When obtaining the stationary solution the equation for the grid was solved in common with the system of Navier–Stokes equations. In the dimensionless form the density and the velocity were normalized on their values in free stream and the temperature—on $c_V U_\infty^2$. The radius of sphere bluntness was chosen as a characteristic dimension. Computations were carried out for different parameters of the free stream. Below some calculation results on the flow past the composite body at the following parameters are carried out: $M_\infty = 8$, $\mathrm{Re}_\infty = 10^4$, $\gamma = 1.4$, $\omega = 0.75$, $\mathrm{Pr} = 0.72$, $a = 0.5$. Stationary solution was determined by using the relaxation method. The iteration step τ was chosen changeable increasing over 50–70 steps from $\tau = h/2$ up to $\tau \approx 4h$ where h is the mesh step in the transverse direction on the symmetry axis (on the uniform grid). The equation for the grid was solved at the following parameters $\alpha = 1.0$, $\tau_1 = \tau$, $\delta = 0.2$. The difference grid contained 25 mesh points in η direction and 18 mesh points in ξ direction. $200 \div 300$ iterations are required for obtaining the stationary solution. At initial moment the undisturbed flow was set in the computed region and the boundary-value conditions were set on the body. In the region Q_2 the boundary-value conditions were transfered on the whole region at $x = 0.0$. In Fig. 1 the computed grid is presented in the case of obtaining the stationary solution in the region Q_1. A sharp condensation of mesh points occurs in a zone of detached shock wave and in the boundary layer.

Parameter distributions on the axis of symmetry in the region Q_1 is presented in Fig. 2. Note, that the relatively small number of mesh points enables one to calculate the shock

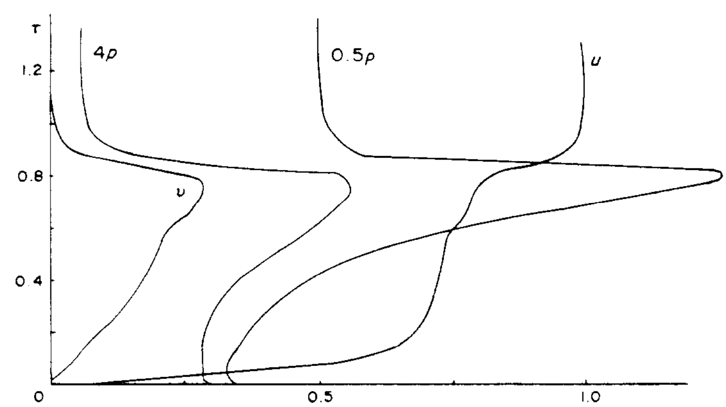

Fig. 4.

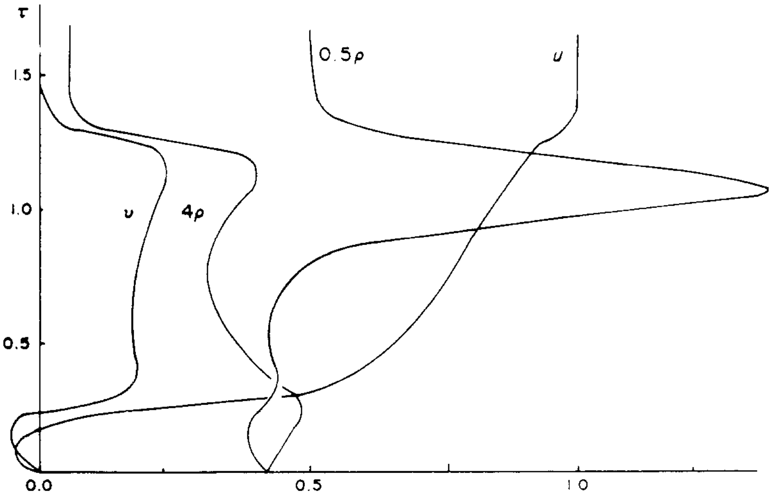

Fig. 5.

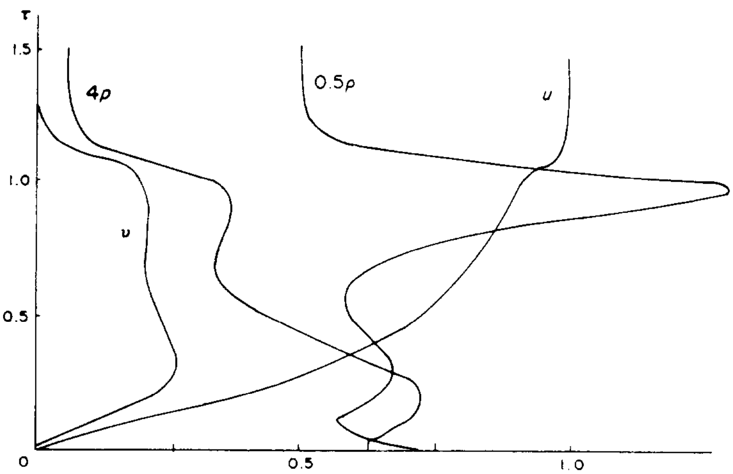

Fig. 6.

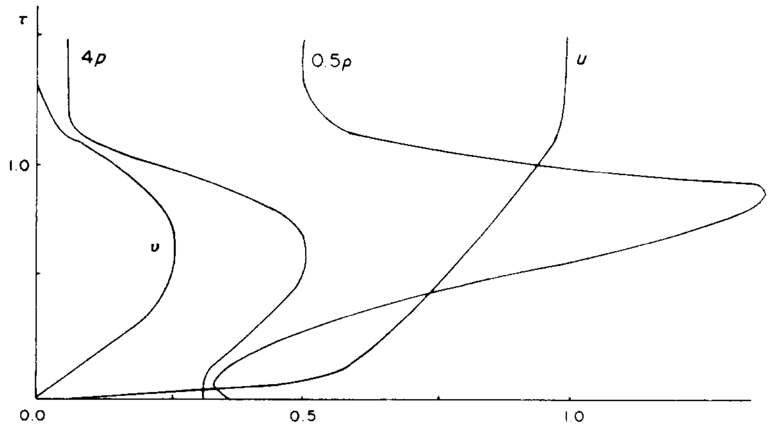

Fig. 7.

transition region and the boundary layer with sufficient accuracy. The computed region Q_2 is presented in Fig. 3. The flow parameters in cross-sections $I(x = 0.4197)$, $II(x = 1.4697)$, $III(x = 1.8697)$, $IV(x = 2.9358)$ are presented in Figs. 4–7. The separated flow which forms a sheaf of compression waves appears in the neighbourhood of the cone (the velocity components change the sign, Fig. 5). In Fig. 6 there are two peaks of gas dynamic values corresponding to the compression waves. Downstream the compression waves interacting with a bow wave interflow. A complex flow occurs downstream on the cylinder, however the grid resolving power has proved to be insufficient for the detailed flow description.

It is instructive to note in conclusion that the proposed method of solving the Navier–Stokes equations on moving grids allows to increase considerably the computation accuracy.

REFERENCES

1. R. D. Richtmayer and K. W. Morton, *Difference Methods for Initial-Value Problems.* Interscience, New York (1967).
2. P. I. Roache, *Computational Fluid Dynamics.* Hermosa, Albuquerque, New Mexico (1972).
3. F. H. Harlow, The particle-in-cell method for fluid dynamics, *Methods in Computational Physics*, Vol. 3. Academic Press, New York (1964).
4. V. I. Polezhaev, Chislennoje reshenoje sistemy dvumernykh nestatsionarnkh uravnenij Nav'je–Stoksa dyla szhimaemogo gaza. *Izv. AN SSSR, ser. mekh. zhid. i gaza*, No. 6 (1966).
5. A. I. Tolstykh, O metodakh chislennogo resheniya uravnenij nav'je-stoksa szhimaemogo gaza v shirokom diapazone chisel rejnol'dsa. *Dokl. Akad. SSSR* **210**(1) (1973).
6. L. I. Severinov, Sposob resheniya nelinejnykh raznostnykh kraevykh zadach mekhaniki sploshnoj sredy. *Zhurnal Byschisl. Matem. i Mat. Fiziki* **18**(4) (1978).
7. V. M. Kovenya, Application of implicit difference schemes to the solution of aerodynamic problems. *Lecture Notes in Physics* **58** (1976).
8. N. N. Yanenko and V. M. Kovenya, A difference method for solving the multidimensional equations of gas dynamics. *Sovjet Math. Dokl.* **18**(1) (1977).
9. W. R. Briley and H. McDonald, Solution of the multidimensional compressible Navier-Stokes equations by a generalized implicit method. *J. Comput. Phys.* **24**(4) (1977).
10. N. N. Yanenko, *The Method of Fractional Steps.* "Nauka", Norosibirsk (1967); English transl., Springer-Verlag, Berlin (1971).
11. A. I. Tolstykh, Ob Issledovanii Techenij Vyazkogo Szhimaemogo Gaza pri Pomoshchi polnykh Uravnenij Nav'je–Stoksa. In *Chislennyje Metody Mekh. Sploshnoj Sredy, Novosibirsk* **6**(4) (1975).
12. S. Burstein and A. Mirin, Third order difference methods for hyperbolic equations. *Lecture Notes in Physics* **8** (1971).
13. N. N. Yanenko, N. T. Danaev and V. D. Lisejkin, O Variatsionnom Metode Postrojeniya Setok. In *Chislennyje Metody Mekhaniki Sploshnoj Sredy, Novosibirsk* **8**(4) (1977).